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5.6.4 The Beta Distribution

A random variable is said to have a beta distribution if its density is given by

$$f(x) = \begin{cases} \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1} & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

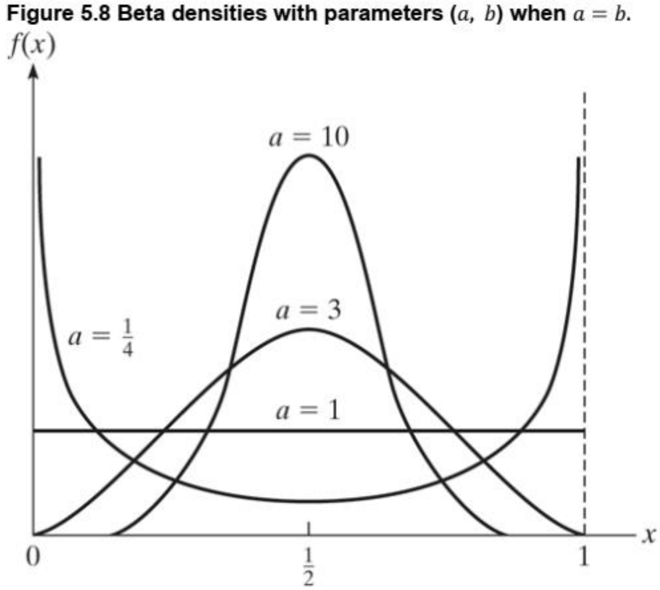
where

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

The beta distribution can be used to model a random phenomenon whose set of possible values is some finite interval $[c,\ d]$—which, by letting c denote the origin and taking $d-c$ as a unit measurement, can be transformed into the interval [0, 1].

When $a=b,$ the beta density is symmetric about $\frac{1}{2},$ giving more and more weight to regions about $\frac{1}{2}$ as the common value a increases. When $a=b=1$, the beta distribution reduces to the uniform $(0,1)$ distribution. (See **Figure 5.8** .) When $b>a,$ the density is skewed to the left (in the sense that smaller values become more likely), and it is skewed to the right when $a>b$. (See **Figure 5.9** .)

Figure 5.8 Beta densities with parameters ($a,\ b$) when $a=b$.



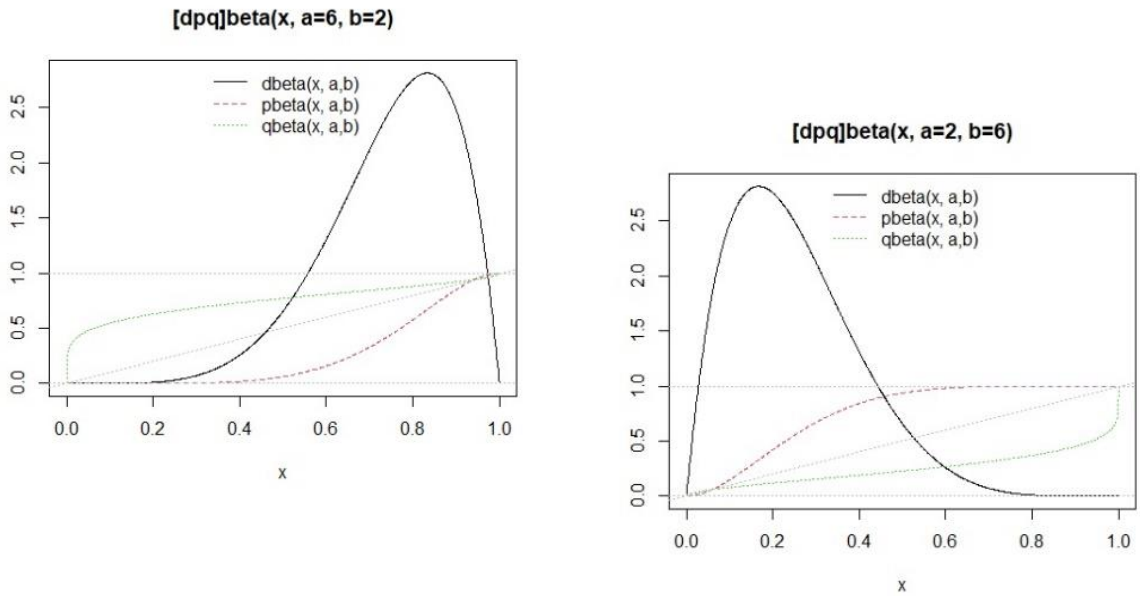
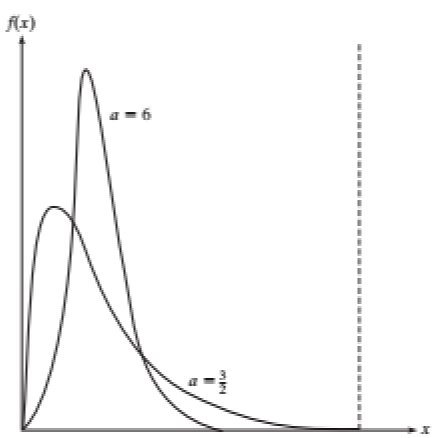


Figure 5.9 Beta densities with parameters ($a,\ b$) when $a/(a+b) = 1/20$.



The relationship

(6.3)

$$B(a,b) = \frac{\Gamma\,(a)\,\Gamma\,(b)}{\Gamma\,(a+b)}$$

can be shown to exist between

$$B(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}\,dx$$

and the gamma function.

Using **Equation (6.3)** along with the identity $\Gamma\,(x+1) = x\,\Gamma\,(x)$, which was given in **Equation (6.1)** it follows that

$$\frac{B(a+1,b)}{B(a,b)} = \frac{\Gamma\,(a+1)\,\Gamma\,(b)}{\Gamma\,(a+b+1)}\,\frac{\Gamma\,(a+b)}{\Gamma\,(a)\,\Gamma\,(b)} = \frac{a}{a+b}$$

The preceding enables us to easily derive the mean and variance of a beta random variable with parameters a and b. For if X is such a random variable, then

$$E[X] = \frac{1}{B(a,b)} \int_0^1 x^a (1-x)^{b-1} \, dx$$

$$= \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{a}{a+b}$$

Similarly, it follows that

$$E[X^2] = \frac{1}{B(a,b)} \int_0^1 x^{a+1} (1-x)^{b-1} \, dx$$

$$= \frac{B(a+2,b)}{B(a,b)}$$

$$= \frac{B(a+2,b)}{B(a+1,b)} \frac{B(a+1,b)}{B(a,b)}$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)}$$

The identity $\mathrm{Var}(X) = E[X^2] - (E[X])^2$ now yields

$$\mathrm{Var}(X) = \frac{a(a+1)}{(a+b)(a+b+1)} - \left(\frac{a}{a+b}\right)^2$$

$$= \frac{ab}{(a+b)^2(a+b+1)}$$

Remark A verification of **Equation (6.3)** appears in **Example 7c** of **Chapter 6**.